

# MONOIDS $\text{Mon}\langle a, b : a^\alpha b^\beta a^\gamma b^\delta a^\varepsilon b^\varphi = b \rangle$ ADMIT FINITE COMPLETE REWRITING SYSTEMS

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**ABSTRACT.** The aim of this note is to prove that monoids  $\text{Mon}\langle a, b : aUb = b \rangle$ , with  $aUb$  of relative length 6, admit finite complete rewriting systems. This is some advance in the understanding the long-standing open problem whether the word problem for one-relator monoids is soluble.

## 1. INTRODUCTION

The word problem for one-relator semigroups is still an open problem. The biggest advance in this is due to the school in Moscow established by Sergei Adian. Adian and his students proved that the word problem for one-relator semigroups can be reduced to the cases  $\text{Mon}\langle a, b : aUb = bVb \rangle$  and  $\text{Mon}\langle a, b : aUb = b \rangle$ . The good reference to their results is the survey paper [1]. The papers of Kobayashi [5] and Zhang [6] prove how powerful are the rewriting systems to work with one-relator monoids. In particular there is an open problem which naturally arises from the work of Kobayashi:

**Open Problem 1.1.** Does every one-relator monoid admits a finite complete rewriting system?

For more details about rewriting systems we refer the reader to [2].

Our objective in this paper is to continue the work started in [3], in which we prove that every monoid  $\text{Mon}\langle a, b : a^\alpha b^\beta a^\gamma b^\delta = b \rangle$  admits a finite complete rewriting system. The proof of this results is quite easy unlike our main result of this note where we consider the monoids  $\text{Mon}\langle a, b : aUb = b \rangle$  with  $aUb$  of relative length 6.

Most importantly, at the end of the note we put some conjectures which might shed some light on how to attack further the word problem for one-relator monoids.

## 2. MAIN RESULT

**Main Theorem.** Every monoid  $\text{Mon}\langle a, b : a^\alpha b^\beta a^\gamma b^\delta a^\varepsilon b^\varphi = b \rangle$  admits a finite complete system.

*Proof.* We may assume that  $a^\alpha b^\beta a^\gamma b^\delta a^\varepsilon b^\varphi$  has overlaps with itself. Consider first the case when the overlap includes  $b^\delta a^\varepsilon$ .

**Case 1:**  $a^\alpha b^\beta a^\gamma b^\delta a^\varepsilon b^\varphi = a^p b^{q+s} a^{r+pk} b^{q+s} a^{r+pk} b^s$  for some  $p, s, k \geq 1, q \geq 0$  and  $0 \leq r < p$

We will consider the following four subcases:

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**Subcase 1a:**  $s = 1$ 

The required finite complete system for our monoid is as follows:

$$\begin{aligned} a^p b^{q+1} a^{r+pk} b^{q+1} a^{r+pk} b &\rightarrow b \\ a^p b^{q+1} a^{r+pi} b &\rightarrow b^{q+1} a^{r+p(i+1)} b, \quad 0 \leq i \leq k-1. \end{aligned}$$

**Subcase 1b:**  $s > 1$  and  $r > 0$ 

Then the monoid admits the following finite complete system:

$$\begin{aligned} a^p b^{q+s} a^{r+pk} b^{q+s} a^{r+pk} b^s &\rightarrow b \\ a^p b^{q+s} a^{r+pi} b &\rightarrow b^{q+1} a^{r+pk} b^s (b^{q+s-1} a^{r+pk} b^{q+s} a^{r+pk} b^s)^{k-1-i}, \quad 0 \leq i \leq k-1. \end{aligned}$$

**Subcase 1c:**  $s > 1$ ,  $r = 0$  and  $k = 1$ 

$M$  admits the following finite complete system:

$$\begin{aligned} a^p b^s &\rightarrow x \\ x b^q x b^q x &\rightarrow b \\ x b^{q+1} &\rightarrow b^{q+1} x. \end{aligned}$$

**Subcase 1d:**  $s > 1$ ,  $r = 0$  and  $k \geq 2$ 

Our relation reads as  $a^p b^{q+s} a^{pk} b^{q+s} a^{pk} b^s = b$ . We add a new letter  $x = a^{pk} b^s$ . Then  $a^p b^{q+s} x b^q x = b$ .

Also:  $a^{p(k-1)} b = a^{pk} b^{q+s} x b^q x = x b^q x b^q x$ . In particular,  $\underline{a^p x b^q x b^q x b^{s-1}} = a^{pk} b^s = \underline{x}$ . Since  $a^p b \cdot b^{q+s-1} x b^q x = b$ , we have that  $\underline{a^p b = x b^q x b^q x (b^{q+s-1} x b^q x)^{k-2}}$ . This yields

$$\begin{aligned} \underline{b} = a^p b^{q+s} x b^q x &= x b^q x b^q x (b^{q+s-1} x b^q x)^{k-2} \cdot b^{q+s-1} x b^q x \\ &= \underline{x b^q x b^q x (b^{q+s-1} x b^q x)^{k-1}}. \end{aligned}$$

The last relation implies the following two consequences:

$$\begin{aligned} \underline{x b^q x b^q x (b^{q+s-1} x b^q x)^{k-2} b^{q+s}} &= x b^q x b^q x (b^{q+s-1} x b^q x)^{k-2} b^{q+s-1} \cdot b = \\ &= x b^q x b^q x (b^{q+s-1} x b^q x)^{k-2} b^{q+s-1} \cdot x b^q x b^q x (b^{q+s-1} x b^q x)^{k-1} = \\ &= x b^q x b^q x (b^{q+s-1} x b^q x)^{k-1} \cdot b^q x (b^{q+s-1} x b^q x)^{k-1} = \underline{b^{q+1} x (b^{q+s-1} x b^q x)^{k-1}} \end{aligned}$$

and

$$\begin{aligned} \underline{x b^q x b^q x (b^{q+s-1} x b^q x)^{k-2} b^{q+s-1} x b^{q+1}} &= \\ &= x b^q x b^q x (b^{q+s-1} x b^q x)^{k-2} b^{q+s-1} x b^q \cdot x b^q x b^q x (b^{q+s-1} x b^q x)^{k-1} = \\ &= x b^q x b^q x (b^{q+s-1} x b^q x)^{k-1} \cdot b^q x b^q x (b^{q+s-1} x b^q x)^{k-1} = \underline{b^{q+1} x b^q x (b^{q+s-1} x b^q x)^{k-1}}. \end{aligned}$$

The underlined relations give us the following rewriting system, defining  $M$ :

$$\begin{aligned}
a^p x b^q x b^q x b^{s-1} &\rightarrow x \\
a^p b &\rightarrow x b^q x b^q x (b^{q+s-1} x b^q x)^{k-2} \\
x b^q x b^q x (b^{q+s-1} x b^q x)^{k-1} &\rightarrow b \\
x b^q x b^q x (b^{q+s-1} x b^q x)^{k-2} b^{q+s} &\rightarrow b^{q+1} x (b^{q+s-1} x b^q x)^{k-1} \\
x b^q x b^q x (b^{q+s-1} x b^q x)^{k-2} b^{q+s-1} x b^{q+1} &\rightarrow b^{q+1} x b^q x (b^{q+s-1} x b^q x)^{k-1}.
\end{aligned}$$

If  $q < s - 1$ , one readily checks that this system is confluent and noetherian (regardless whether  $q > 2$  or  $q = 2$ ).

If  $q \geq s - 1$ , then

$$\begin{aligned}
a^p x b^{q+1} &= x b^{q-(s-1)} x (b^{q+s-1} x b^q x)^{k-1} \\
a^p x b^q x b^{q+1} &= x b^{q-(s-1)} x b^q x (b^{q+s-1} x b^q x)^{k-1}
\end{aligned}$$

and adding the rules

$$\begin{aligned}
a^p x b^{q+1} &\rightarrow x b^{q-(s-1)} x (b^{q+s-1} x b^q x)^{k-1} \\
a^p x b^q x b^{q+1} &\rightarrow x b^{q-(s-1)} x b^q x (b^{q+s-1} x b^q x)^{k-1}
\end{aligned}$$

to the system, we obtain the required finite complete system.

**Case 2:**  $a^\alpha b^\beta a^\gamma b^\delta a^\varepsilon b^\varphi = a^p b^{q+s} a^\gamma b^\delta a^{r+pk} b^s$  for some  $p, s, k \geq 1, q \geq 0, 0 \leq r < p$  and  $\gamma, \delta \geq 1$  such that either  $\gamma \neq r + pk$  or  $\delta \neq q + s$

**Subcase 2a:**  $r > 0$  and  $s = 1$

The conditions of this subcase translate as  $a^p b^{q+1} a^\gamma b^\delta a^{pk+r} b = b$ . We can turn this to the following equivalent system:

$$\begin{aligned}
a^p b^{q+1} a^\gamma b^\delta a^{pk+r} b &\rightarrow b \\
a^p b^{q+1} a^\gamma b^\delta a^{pi+r} b &\rightarrow b^{q+1} a^\gamma b^\delta a^{p(i+1)+r} b, \quad 0 \leq i \leq k-1.
\end{aligned}$$

If  $\gamma \notin \{p+r, \dots, p(k-1)+r\}$  or  $\delta \neq q+1$ , then this system is complete. So let us assume that  $\delta = q+1$  and  $\gamma = px+r$  where  $1 \leq x \leq k-1$ . Then our obtained system translates as

$$\begin{aligned}
a^p b^{q+1} a^{px+r} b^{q+1} a^{pk+r} b &\rightarrow b \\
a^p b^{q+1} a^{px+r} b^{q+1} a^{pi+r} b &\rightarrow b^{q+1} a^{px+r} b^{q+1} a^{p(i+1)+r} b, \quad 0 \leq i \leq k-1,
\end{aligned}$$

which has a particular rule  $a^p b^{q+1} a^{px+r} b^{q+1} a^{px+r} b \rightarrow b^{q+1} a^{px+r} b^{q+1} a^{p(x+1)+r} b$ . Overlapping it with the first rule, we get a new rule

$$a^p b^{q+1} a^{p(x-1)+r} b \rightarrow b^{q+1} a^{px+r} b^{q+1} a^{p(x+1)+r} b^{q+1} a^{pk+r} b.$$

Overlapping it consecutively with the first rule, we obtain:

$$a^p b^{q+1} a^{pj+r} b \rightarrow b^{q+1} a^{px+r} b^{q+1} a^{p(x+1)+r} b^{q+1} a^{p(j+k-x+1)+r} b, \quad 0 \leq j \leq x-1.$$

Some of the previous rules become consequences of these newly obtained, and one checks that the following system is complete:

$$\begin{aligned}
a^p b^{q+1} a^{px+r} b^{q+1} a^{pk+r} b &\rightarrow b \\
a^p b^{q+1} a^{px+r} b^{q+1} a^{pi+r} b &\rightarrow b^{q+1} a^{px+r} b^{q+1} a^{p(i+1)+r} b, \quad x \leq i \leq k-1 \\
a^p b^{q+1} a^{pj+r} b &\rightarrow b^{q+1} a^{px+r} b^{q+1} a^{p(x+1)+r} b^{q+1} a^{p(j+k-x+1)+r} b, \quad 0 \leq j \leq x-1.
\end{aligned}$$

**Subcase 2b:  $r > 0$  and  $s > 1$** 

As in **Subcase 2a**, our system generates

$$\begin{aligned} a^p b^{q+s} a^\gamma b^\delta a^{pk+r} b^s &\rightarrow b \\ a^p b^{q+s} a^\gamma b^\delta a^{pi+r} b &\rightarrow b^{q+1} a^\gamma b^\delta a^{pk+r} b^s (b^{q+s-1} a^\gamma b^\delta a^{pk+r} b^s)^{k-1-i}, \quad 0 \leq i \leq k-1. \end{aligned}$$

If  $\gamma \notin \{p+r, \dots, p(k-1)+r\}$  or  $\delta \neq q+s$ , then this system is complete. So, let  $\delta = q+s$  and  $\gamma = px+r$  for some  $1 \leq x \leq k-1$ . Then our system can be turned to the following complete system:

$$\begin{aligned} a^p b^{q+s} a^{px+r} b^{q+s} a^{pk+r} b^s &\rightarrow b \\ a^p b^{q+s} a^{px+r} b^{q+s} a^{pi+r} b &\rightarrow b^{q+1} a^{px+r} b^{q+s} a^{pk+r} b^s (b^{q+s-1} a^{px+r} b^{q+s} a^{pk+r} b^s)^{k-1-i}, \quad x \leq i \leq k-1 \\ a^p b^{q+s} a^{pj+r} b &\rightarrow b^{q+1} a^{px+r} b^{q+s} a^{pk+r} b^s (b^{q+s-1} a^{px+r} b^{q+s} a^{pk+r} b^s)^{k-1-x} . \\ &\quad b^{q+s-1} a^{pk+r} b^s (b^{q+s-1} a^{px+r} b^{q+s} a^{pk+r} b^s)^{x-1-j}, \quad 0 \leq j \leq x-1. \end{aligned}$$

**Subcase 2c:  $r = 0$  and  $k > 1$** 

Letting  $a^p b^{q+s} = x$ ,  $a^\gamma b^\delta = y$  and  $a^{pk} b^s = z$ , together with the rule  $xyz \rightarrow b$ , we get the following equivalent system:

$$\begin{aligned} a^\gamma b^\delta &\rightarrow y \\ a^p z b^q y z b^{s-1} &\rightarrow z \\ a^p x &\rightarrow z b^q (y z b^{q+s-1})^{k-2} \\ a^p b &\rightarrow z b^q (y z b^{q+s-1})^{k-2} y z \\ xyz &\rightarrow b \\ z b^q (y z b^{q+s-1})^{k-1} &\rightarrow x \\ xyx &\rightarrow b^{q+1} (y z b^{q+s-1})^{k-1} \\ xyb &\rightarrow b^{q+1} (y z b^{q+s-1})^{k-1} y z. \end{aligned}$$

If  $q < s-1$ , then this system is complete. If  $q \geq s-1$ , then we add to this system two rules

$$\begin{aligned} a^p z b^q y x &\rightarrow z b^{q-(s-1)} (y z b^{q+s-1})^{k-1} \\ a^p z b^q y b &\rightarrow z b^{q-(s-1)} (y z b^{q+s-1})^{k-1} y z, \end{aligned}$$

and the new system becomes complete.

**Subcase 2d:  $r = 0$  and  $k = 1$** 

Letting  $a^p b^s = x$ , we obtain the following equivalent system:

$$\begin{aligned} a^p b^s &\rightarrow x \\ x b^q a^\gamma b^\delta x &\rightarrow b \\ x b^q a^\gamma b^{\delta+1} &\rightarrow b^{q+1} a^\gamma b^\delta x. \end{aligned}$$

If  $\gamma < p$ , then the system is complete. So, we may assume that  $\gamma = pt + u$  where  $t \geq 1$  and  $0 \leq u < p$ .

Consider first the case when  $u \neq 0$ . If  $\delta < s$ , then we get the following complete system:

$$\begin{aligned} a^p b^s &\rightarrow x \\ x b^q a^{pt+u} b^\delta x &\rightarrow b \\ x b^q a^{pt+u} b^{\delta+1} &\rightarrow b^{q+1} a^{pt+u} b^\delta x \\ x b^q a^{pi+u} x &\rightarrow b^{q+1} a^{pt+u} b^\delta x b^{s-(\delta+1)} (b^q a^{pt+u} b^\delta x b^{s-1})^{t-1-i}, \quad 0 \leq i \leq t-1 \\ x b^q a^{pi+u} b &\rightarrow b^{q+1} a^{pt+u} b^\delta x b^{s-(\delta+1)} (b^q a^{pt+u} b^\delta x b^{s-1})^{t-1-i} b^q a^{pt+u} b^\delta x, \quad 0 \leq i \leq t-1. \end{aligned}$$

If  $\delta \geq s$ , then we get the following system:

$$\begin{aligned} a^p b^s &\rightarrow x \\ x b^q a^{p(t-1)+u} x b^{\delta-s} x &\rightarrow b \\ x b^q a^{p(t-1)+u} x b^{\delta+1-s} &\rightarrow b^{q+1} a^{p(t-1)+u} x b^{\delta-s} x. \end{aligned}$$

If  $q < \delta + 1 - s$ , then this system is complete. If  $q \geq \delta + 1 - s$ , then we add the following rules to the system and the resulting system will be complete:

$$\begin{aligned} x b^q a^{pi+u} b &\rightarrow b^{q+1} a^{p(t-1)+u} x b^{\delta-s} x b^{q-(\delta+1-s)} a^{p(t-1)+u} x b^{\delta-s} x \\ &\quad (b^{q+s-1} a^{p(t-1)+u} x b^{\delta-s} x)^{t-1-i}, \quad 0 \leq i \leq t-1 \\ x b^q a^{pi+u} x &\rightarrow b^{q+1} a^{p(t-1)+u} x b^{\delta-s} x b^{q-(\delta+1-s)} a^{p(t-1)+u} x b^{\delta-s} x \\ &\quad (b^{q+s-1} a^{p(t-1)+u} x b^{\delta-s} x)^{t-1-i} b^{s-1}, \quad 0 \leq i \leq t-1. \end{aligned}$$

We are left with the case when  $u = 0$ . First consider the case when  $t \geq 2$ . We have the following equivalent system (with a new  $x$ ):

$$\begin{aligned} a^{pt} b^\delta a^p b^s &\rightarrow x \\ a^p b^{q+s} x &\rightarrow b \\ a^{pt} b^{\delta+1} &\rightarrow x b^q x. \end{aligned}$$

If  $\delta \geq q + s$ , then this system is complete. If  $\delta \leq q + s - 1$ , then we get the following complete system:

$$\begin{aligned} a^p b &\rightarrow x b^q x b^{q+s-(\delta+1)} (b^{q+s-1} x)^{t-1} \\ a^p x b^q x b^{q+s-2} x b^q x b^{q+s-(\delta+1)} (b^{q+s-1} x)^{t-1} b^{s-1} &\rightarrow x \\ x b^q x b^{q+s-(\delta+1)} (b^{q+s-1} x)^{t-1} b^{q+s-1} x &\rightarrow b \\ a^p x b^q x b^{q+s-1} &\rightarrow x b^q x. \end{aligned}$$

Finally, let  $t = 1$ . If  $\delta < s$ , then we obtain the following complete system:

$$\begin{aligned} a^p b^s &\rightarrow x \\ x b^q a^p b^\delta x &\rightarrow b \\ x b^q a^p b^{\delta+1} &\rightarrow b^{q+1} a^p b^\delta x \\ x b^{q+s-(\delta+1)} x &\rightarrow b^{q+1} a^p b^\delta x b^{s-(\delta+1)} \\ x b^{q+s-\delta} &\rightarrow b^{q+1} a^p b^\delta x b^{q+s-(\delta+1)} a^p b^\delta x. \end{aligned}$$

Let  $\delta \geq s$ . We get the following system:

$$\begin{aligned} a^p b^s &\rightarrow x \\ x b^q x b^{\delta-s} x &\rightarrow b. \end{aligned}$$

Recall that  $\delta - s \neq q$ . If  $q < \delta - s + 1$ , then we obtain the following complete system:

$$\begin{aligned} a^p b^s &\rightarrow x \\ x b^q x b^{\delta-s} x &\rightarrow b \\ x b^q x b^{\delta-s+1} &\rightarrow b^{q+1} x b^{\delta-s} x. \end{aligned}$$

If  $q \geq \delta - s + 1$ , then we add to this system the rule

$$x b^{q+1} \rightarrow b^{q+1} x b^{\delta-s} x b^{q-(\delta-s+1)}$$

and the system becomes complete.  $\square$

### 3. CONCLUDING REMARKS

Carefully observing the cases in the proof of our main result and the cases from the proof of the main result in [3], we noticed that each monoid under consideration has at most quadratic Dehn function and linear space function, so we raise the following

**Open Problem 3.1.** Is it true that every monoid  $\text{Mon}\langle a, b : aUb = b \rangle$  has at most quadratic Dehn function and linear space function?

We should note that using the method of word diagrams, it can be quite problematic for the monoids  $\text{Mon}\langle a, b : aUb = b \rangle$  to find the corresponding Dehn or space functions. The reason is that if two words  $u$  and  $v$  over  $\{a, b\}$  are equal in  $\text{Mon}\langle a, b : aUb = b \rangle$ , then there exists a word  $w \in \{a, b\}^+$  which can be obtained from  $u$  only by successive insertions  $b \rightarrow aUb$ , and so that  $v$  can be obtained from  $w$  only by successive deletions  $aUb \rightarrow b$ . Even though, on one hand this makes the diagrams look quite easy, on the other hand it is not clear how to unfold the regions touching the longest path in the diagrams.

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